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ON A FUNCTIONAL EQUATION CONNECTED TO GAUSS QUADRATURE RULE

BARBARA KOCŁĘGA-KULPA, TOMASZ SZOSTOK

Abstract. We consider the functional equation

$$F(y) - F(x) = (y - x)[f(\alpha x + \beta y) + f(\beta x + \alpha y)]$$

stemming from Gauss quadrature rule. In previous results equations of this type with rational only coefficients α and β were considered. In this paper we allow these numbers to be irrational. We find all solutions of this equation for functions acting on \mathbb{R} . However, some results are valid also on integral domains.

1. Introduction

Functional equations connected to well known quadrature rules were studied by many authors. In the monograph [7] the equation stemming from Simpson quadrature rule

$$F(y) - F(x) = (y - x) \left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y) \right]$$

was considered for functions acting on \mathbb{R} . This equation, in a bit more general form, for functions transforming an integral domain into itself has been solved in [3].

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On the other hand, M. Sablik [9] during the 7th Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities presented the general solution of the equation

$$(1) \quad g(x) - f(y) = (x - y)[h(x) + k(sx + ty) + k(tx + sy) + h(y)]$$

in the case $s, t \in \mathbb{Q}$ without any regularity assumptions concerning the functions considered. Some other equations of this type were solved in [4]. However, these results are valid only for equations with rational coefficients on the right-hand side.

In the current paper we are going to consider an equation connected to the simplest quadrature rule with irrational coefficient - the Gauss quadrature rule with two nodes. Thus we shall find the solutions of

$$(2) \quad F(y) - F(x) = (y - x)[f(\alpha x + \beta y) + f(\beta x + \alpha y)].$$

In our method we shall need a lemma, which was established by M. Sablik ([8, Lemma 2.3]) and improved by I. Pawlikowska (cf. [6]).

By a *polynomial function of order n* in this paper we mean a solution of the functional equation $\Delta_h^{n+1}f(x) = 0$, where Δ_h^n stands for the n -th iterate of the difference operator $\Delta_h f(x) = f(x + h) - f(x)$. Observe that a continuous polynomial function of order n is a polynomial of degree at most n (see [5, Theorem 4, p. 398]).

Let us now quote Sablik's result. First we need some notations. Let G, H be Abelian groups and $SA^0(G, H) := H$, $SA^1(G, H) := \text{Hom}(G, H)$ (i.e. the group of all homomorphisms from G into H), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G, H)$ be the group of all i -additive and symmetric mappings from G^i into H . Furthermore, let $\mathcal{P} := \{(\alpha, \beta) \in \text{Hom}(G, G)^2 : \alpha(G) \subset \beta(G)\}$. Finally, for $x \in G$ let $x^i = \underbrace{(x, \dots, x)}_i$, $i \in \mathbb{N}$.

LEMMA 1. Fix $N \in \mathbb{N} \cup \{0\}$ and let I_0, \dots, I_N be finite subsets of \mathcal{P} . Suppose that H is uniquely divisible by $N!$ and that the functions $\varphi_i: G \rightarrow SA^i(G, H)$ and $\psi_{i,(\alpha,\beta)}: G \rightarrow SA^i(G, H)$, $(\alpha, \beta) \in I_i$, $i = 0, \dots, N$, satisfy

$$(3) \quad \varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^N \sum_{(\alpha,\beta) \in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x) + \beta(y))(y^i)$$

for every $x, y \in G$. Then φ_N is a polynomial function of order at most $k - 1$, where

$$k = \sum_{i=0}^N \text{card} \left(\bigcup_{s=i}^N I_s \right).$$

2. Results

First we need some auxiliary lemma.

LEMMA 2. Let P be an integral domain with a unit element 1 such that $2 = 1 + 1 \neq 0$ and $3 = 1 + 1 + 1 \neq 0$. Let the functions $f, F: P \rightarrow P$ satisfy the equation

$$(4) \quad F(x) - F(y) = (x - y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)]$$

for all $x, y \in P$ and some $b_1, \dots, b_n, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in P$. Further, let f be a function of the form

$$(5) \quad f(x) = a_3(x) + a_2(x) + a_1(x) + a_0, \quad x \in P,$$

where $a_i, i \in \{1, 2, 3\}$ is a diagonalization of some i -additive and symmetric function and $a_0 \in P$ is a constant.

Then all functions $a_i, i \in \{1, 2, 3\}$ also satisfy (4) with some F_i .

PROOF. We may assume that $F(0) = 0$, so substituting $y = 0$ in (4) we obtain

$$F(x) = x[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)], \quad x \in P.$$

Using this formula in (4) we get

$$(6) \quad x[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)] - y[b_1 f(\alpha_1 y) + \cdots + b_n f(\alpha_n y)] \\ = (x - y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)], \quad x, y \in P.$$

It is easy to see that if f, g are solutions to this equation, then also $f + g$ satisfies this equation. Moreover, if f satisfies (6) and $\alpha \in P$ is given, then also αf is a solution. On the other hand, if αf satisfies (6) and $\alpha \neq 0$, then f

is a solution of this equation. Now take $a \neq 0$ and substitute ax, ay in place of x and y . We obtain (after canceling by a) that

$$\begin{aligned} & x[b_1f(\alpha_1ax) + \cdots + b_nf(\alpha_nax)] - y[b_1f(\alpha_1ay) + \cdots + b_nf(\alpha_nay)] \\ &= (x - y)[b_1f(\alpha_1ax + \beta_1ay) + \cdots + b_nf(\alpha_nax + \beta_nay)], \quad x, y \in P, \end{aligned}$$

which means that the function $x \mapsto f(ax)$ also satisfies our equation (if f does).

Now consider f of the form (5), and put

$$g_1(x) := 8f(x) - f(2x), \quad x \in P.$$

Then g_1 is a solution of (6) and on the other hand

$$(7) \quad g_1(x) = 4a_2(x) + 6a_1(x) + 7a_0.$$

Define g_2 by the formula

$$g_2(x) := 4g_1(x) - g_1(2x) = 12a_1(x) + 21a_0.$$

Similarly as before g_2 satisfies (6), constant function $21a_0$ is clearly a solution of (6), which means that also $12a_1$ satisfies this equation thus also a_1 satisfies (6) since $2, 3 \neq 0$. Further, in view of (7) we easily obtain that also a_2 is a solution of (6). Finally, from (5) it follows that a_3 satisfies our equation. \square

REMARK 1. Let P be an integral domain such that $2, 3 \neq 0$. If functions $A_i, B_i: P \rightarrow P$, $i = 1, 2, 3$, satisfy

$$(8) \quad A_3(x) + A_2(x) + A_1(x) = B_3(x) + B_2(x) + B_1(x), \quad x \in P,$$

and A_i, B_i are i -homogeneous, then

$$A_i = B_i, \quad i = 1, 2, 3.$$

Indeed, if we take $-x$ in place of x , in (8), then we obtain

$$(9) \quad -A_3(x) + A_2(x) - A_1(x) = -B_3(x) + B_2(x) - B_1(x), \quad x \in P.$$

This, together with (8) means that $A_2 = B_2$ and, consequently

$$(10) \quad A_3(x) + A_1(x) = B_3(x) + B_1(x), \quad x \in P.$$

Moreover, taking here $2x$ instead of x , we get

$$(11) \quad 8A_3(x) + 2A_1(x) = 8B_3(x) + 2B_1(x), \quad x \in P.$$

Multiplying (10) by 2 and subtracting it from (11) we arrive at

$$6A_3(x) = 6B_3(x), \quad x \in P,$$

which means that $A_3 = B_3$ and obviously also $A_1 = B_1$.

Now we shall prove some lemmas concerning solutions of a simplified version of equation (2). However, at this moment we are concerned only with solutions, which are diagonalizations of symmetric and i -additive functions.

LEMMA 3. *Let P be a integral domain with a unit element 1 such that $2, 3 \neq 0$. If functions $G, f: P \rightarrow P$ satisfy the equation*

$$(12) \quad G(y) - G(x) = (y - x)[f(x + \gamma y) + f(\gamma x + y)], \quad x, y \in P,$$

and $f(x) := C(x, x, x)$ for some 3-additive and symmetric function $C: P \rightarrow P$, then

$$f((\gamma + 1)x) = 2cx^3$$

for some $c \in P$ and all $x \in P$.

PROOF. Let f be given by the formula $f(x) = C(x, x, x)$, $x \in P$, where C is 3-additive and symmetric.

We may assume that $G(0) = 0$, so substituting $x = 0$ in (12) we get

$$(13) \quad G(y) = y[f(\gamma y) + f(y)], \quad y \in P.$$

Using this formula in (12), we obtain for all $x, y \in P$

$$(14) \quad y[f(\gamma y) + f(y)] - x[f(\gamma x) + f(x)] = (y - x)[f(x + \gamma y) + f(\gamma x + y)].$$

On the other hand, using here the form of f after some calculation we arrive at

$$(15) \quad x[f(\gamma y) + f(y)] - y[f(\gamma x) + f(x)] = 3(y - x)[C(x, x, \gamma y) + C(x, \gamma y, \gamma y) + C(y, y, \gamma x) + C(y, \gamma x, \gamma x)]$$

for all $x, y \in P$.

Now we define a new function $g(x) := f(x) + f(\gamma x)$, $x \in P$. Then in view of (15) we have

$$(16) \quad \begin{aligned} xg(y) - yg(x) &= 3(y-x)[C(x, x, \gamma y) + C(x, \gamma y, \gamma y) \\ &\quad + C(y, y, \gamma x) + C(y, \gamma x, \gamma x)]. \end{aligned}$$

Further, taking here $x = 1$ we obtain the following formula

$$(17) \quad g(y) - yg(1) = 3(y-1)[C(1, 1, \gamma y) + C(1, \gamma y, \gamma y) + C(y, y, \gamma) + C(y, \gamma, \gamma)].$$

Using Remark 1, we may compare here the terms of order one and we obtain that

$$(18) \quad yg(1) = 3C(y, \gamma, \gamma) + 3C(1, 1, \gamma y), \quad y \in P,$$

further comparing expressions of order two in (17) we get

$$(19) \quad 3y[C(y, \gamma, \gamma) + C(1, 1, \gamma y)] = 3[C(1, \gamma y, \gamma y) + C(y, y, \gamma)], \quad y \in P,$$

thus, using here (18), we may write

$$(20) \quad y^2g(1) = 3[C(1, \gamma y, \gamma y) + C(y, y, \gamma)], \quad y \in P.$$

Finally, comparing terms of order three in (17), we obtain

$$g(y) = 3y[C(1, \gamma y, \gamma y) + C(y, y, \gamma)], \quad y \in P,$$

which, in view of (20), gives us

$$(21) \quad g(y) = y^3g(1), \quad y \in P,$$

so we may write that

$$f(x) + f(\gamma x) = cx^3, \quad x \in P,$$

(where $c = g(1)$) and, more precisely,

$$(22) \quad C(x, x, x) + C(\gamma x, \gamma x, \gamma x) = cx^3, \quad x \in P.$$

Now, putting $2x$ in place of y in (16), we obtain

$$6cx^4 = 3x[2C(x, x, \gamma x) + 4C(x, \gamma x, \gamma x) + 4C(x, x, \gamma x) + 2C(x, \gamma x, \gamma x)],$$

thus

$$(23) \quad 3C(x, x, \gamma x) + 3C(x, \gamma x, \gamma x) = cx^3, \quad x \in P.$$

Adding equations (22) and (23), we may write

$$C(x, x, x) + 3C(x, x, \gamma x) + 3C(x, \gamma x, \gamma x) + C(\gamma x, \gamma x, \gamma x) = 2cx^3, \quad x \in P,$$

i.e.

$$f((1 + \gamma)x) = 2cx^3, \quad x \in P. \quad \square$$

LEMMA 4. *Let P be an integral domain with a unit element 1 such that $2 \neq 0$. If functions $G, f: P \rightarrow P$ satisfy the equation (12) and $f(x) := B(x, x)$ for some 2-additive and symmetric function $B: P^2 \rightarrow P$, then*

$$2f((\gamma + 1)x) = 3bx^2$$

for some $b \in P$ and all $x \in P$.

PROOF. Let $f(x) = B(x, x)$, $x \in P$, where B is biadditive and symmetric. We may assume, without loss of generality, that $G(0) = 0$ and putting $x = 0$ in (12) we obtain

$$(24) \quad G(y) = y[f(y) + f(\gamma y)], \quad y \in P.$$

Consequently, equation (12) takes form

$$y[f(y) + f(\gamma y)] - x[f(x) + f(\gamma x)] = (y - x)[f(x + \gamma y) + f(\gamma x + y)], \quad x, y \in P.$$

Further, using here the form of f , we may write for all $x, y \in P$

$$\begin{aligned} y[f(y) + f(\gamma y)] - x[f(x) + f(\gamma x)] &= (y - x)[f(x) + 2B(x, \gamma y) + f(\gamma y) \\ &\quad + f(\gamma x) + 2B(\gamma x, y) + f(y)]. \end{aligned}$$

Now we define a new function $g(x) := f(x) + f(\gamma x)$, $x \in P$, then we obtain

$$yg(y) - xg(x) = (y - x)[g(x) + g(y) + 2B(x, \gamma y) + 2B(\gamma x, y)], \quad x, y \in P,$$

and, after some simple calculations,

$$(25) \quad xg(y) - yg(x) = 2(y - x)[B(x, \gamma y) + B(\gamma x, y)], \quad x, y \in P.$$

Putting here $x = 1$ we get

$$g(y) - yg(1) = 2(y - 1)[B(1, \gamma y) + B(\gamma, y)], \quad x, y \in P.$$

Similarly as in the proof of the previous lemma we compare now the terms of the same order. Thus we get

$$g(y) = 2y[B(1, \gamma y) + B(\gamma, y)], \quad y \in P,$$

and

$$yg(1) = 2[B(1, \gamma y) + B(\gamma, y)], \quad y \in P,$$

which means that

$$g(y) = by^2, \quad y \in P,$$

where $b := g(1)$. Further, from the obtained form of g we have

$$f(x) + f(\gamma x) = bx^2, \quad x \in P,$$

i.e.

$$(26) \quad 2B(x, x) + 2B(\gamma x, \gamma x) = 2bx^2, \quad x \in P.$$

On the other hand, taking in (25) $2x$ in place of y , and using form of g , we get

$$4B(x, \gamma x) = bx^2, \quad x \in P.$$

Adding this equality to (26), we may write

$$2B(x, x) + 2B(\gamma x, \gamma x) + 4B(\gamma x, x) = 3bx^2,$$

thus

$$2f((1 + \gamma)x) = 3bx^2, \quad x \in P. \quad \square$$

LEMMA 5. *Let P be an integral domain and let $f, G: P \rightarrow P$ be solutions of (12) such that f is an additive function. Then*

$$f((\gamma + 1)x) = ax$$

for some $a \in P$ and all $x \in P$.

PROOF. Similarly as before we easily obtain that f satisfies (14). Since f is additive, we get

$$y[f(\gamma y) + f(y)] - x[f(\gamma x) + f(x)] = (y - x)[f(x) + f(\gamma y) + f(\gamma x) + f(y)]$$

for all $x, y \in P$ and, consequently,

$$x[f(\gamma y) + f(y)] = y[f(\gamma x) + f(x)], \quad x, y \in P.$$

Taking here $y = 1$ we arrive at

$$x[f(\gamma) + f(1)] = f(\gamma x) + f(x), \quad x \in P,$$

thus putting $a := f(\gamma) + f(1)$ we have

$$f(\gamma x) + f(x) = ax.$$

Using here the additivity of f we obtain $f((\gamma + 1)x) = ax$ for all $x \in P$. \square

Now we are in position to obtain the result concerning the form of solutions of equation (2) for functions acting on an integral domain.

THEOREM 1. *Let P be an integral domain divisible by 6. If functions $f, F : P \rightarrow P$ satisfy (12) with some $\gamma \in P$, such that P is divisible by γ , $\gamma + 1$ and $\gamma - 1$, then*

$$f(x) = cx^3 + bx^2 + ax + d, \quad x \in P,$$

for some $a, b, c, d \in P$.

PROOF. Substituting in (12) $\frac{x-y}{\gamma+1}$ in place of x and $\frac{x+\gamma y}{\gamma+1}$ in place of y , we obtain

$$G\left(\frac{x+\gamma y}{\gamma+1}\right) - G\left(\frac{x-y}{\gamma+1}\right) = y[f((x+(\gamma-1)y) + f(x)], \quad x, y \in P,$$

and, consequently,

$$yf(x) = G\left(\frac{x+\gamma y}{\gamma+1}\right) - G\left(\frac{x-y}{\gamma+1}\right) - yf((x+(\gamma-1)y), \quad x, y \in P.$$

Using here Lemma 1 with $N = 1$, $I_0 := \left\{ \left(\frac{1}{\gamma+1} \text{id}, \frac{\gamma}{\gamma+1} \text{id} \right), \left(\frac{1}{\gamma+1} \text{id}, \frac{-1}{\gamma+1} \text{id} \right) \right\}$ and $I_1 = \{(\text{id}, (\gamma-1)\text{id})\}$ we obtain that f is a polynomial function of degree at most 3. Now, since P is divisible by 6, we have

$$f(x) = C(x, x, x) + B(x, x) + A(x) + d, \quad x \in P,$$

where $C: P^3 \rightarrow P$, $B: P^2 \rightarrow P$, and $A: P \rightarrow P$ are symmetric and 3, 2, 1-additive, respectively (see for example [5]).

Now using Lemma 2 we obtain that functions $f_3(x) := C(x, x, x)$, $f_2(x) := B(x, x)$, and $f_1(x) := A(x)$ also satisfy (12), which in view of Lemmas 3, 4 and 5 means that $f_3((\gamma+1)x) = c_0 x^3$, $f_2((\gamma+1)x) = b_0 x^2$, and $f_1((\gamma+1)x) = a_0 x$ for some $a_0, b_0, c_0 \in P$. However, P is divisible by $\gamma+1$ and, consequently, taking $c := \frac{c_0}{(\gamma+1)^3}$, $b := \frac{b_0}{(\gamma+1)^2}$ and $a := \frac{a_0}{\gamma+1}$, we arrive at

$$f(x) = cx^3 + bx^2 + ax + d, \quad x \in P. \quad \square$$

Now we shall deal with functions defined and taking values in \mathbb{R} . In this case we shall obtain a general solution of equation (2).

THEOREM 2. *Let functions $f, F: \mathbb{R} \rightarrow \mathbb{R}$ satisfy (2) with some constants $\alpha, \beta \in \mathbb{R}$.*

- (i) *If $\alpha = \beta \neq 0$, then $f(x) = ax + d, x \in \mathbb{R}$.*
- (ii) *If $\alpha = \beta = 0$, then f may be any function.*
- (iii) *If $\alpha = 0, \beta \neq 0$ or $\alpha \neq 0, \beta = 0$, then $f(x) = ax + d, x \in \mathbb{R}$.*
- (iv) *If $\alpha = -\beta \neq 0$, then $f(x) = h(x) + \frac{A(x)}{x}$ for all $x \neq 0$ and some functions $h, A: \mathbb{R} \rightarrow \mathbb{R}$ which are odd and additive, respectively.*
- (v) *If $\alpha \neq -\beta$ and $\alpha^2 + \beta^2 \neq 4\alpha\beta$, then $f(x) = ax + d, x \in \mathbb{R}$.*
- (vi) *If $\alpha \neq -\beta$ and $\alpha^2 + \beta^2 = 4\alpha\beta$, then $f(x) = cx^3 + bx^2 + ax + d, x \in \mathbb{R}$.*

In each case, function F is expressed by formula

$$F(x) = x[f(\alpha x) + f(\beta x)] + F(0), \quad x \in \mathbb{R}.$$

Conversely, in each case functions given by the above formulas satisfy equation (2).

PROOF. Let us first consider the case $\alpha = \beta \neq 0$. Then equation (2) takes the form

$$F(y) - F(x) = 2(y - x)f(\alpha(x + y)).$$

Put $\tilde{f}(t) := 2f(\alpha t)$, $t \in \mathbb{R}$. Then we get the equation solved by J. Aczél in [1] and consequently we obtain $\tilde{f}(x) = \tilde{a}x + \tilde{d}$, which means that $f(x) = ax + d$ where $a := \frac{\tilde{a}}{2\alpha}$ and $d := \frac{\tilde{d}}{2}$.

In the case (iii) we get (after a simple substitution)

$$F(y) - F(x) = (y - x)[h(x) + h(y)]$$

and we obtain our result from the paper of Sh. Haruki [2].

Now we consider the case $\alpha = -\beta \neq 0$, this means that equation (2) takes the form

$$F(y) - F(x) = (y - x)[f(\alpha(x - y)) + f(\alpha(y - x))].$$

Define a new function $f_1: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_1(x) := f(\alpha x).$$

We obtain the equation

$$(27) \quad F(y) - F(x) = (y - x)[f_1(x - y) + f_1(y - x)], \quad x, y \in \mathbb{R}.$$

Take in (27) $x = 0$ to obtain

$$(28) \quad F(y) = y[f_1(-y) + f_1(y)] + F(0), \quad y \in \mathbb{R},$$

which used in (27) gives us

$$F(y) - F(x) = F(y - x) - F(0), \quad x, y \in \mathbb{R}.$$

Substituting here $x + y$ in place of y we get

$$F(x + y) - F(x) = F(y) - F(0), \quad x, y \in \mathbb{R}.$$

This means that function $A_1 := F - F(0)$ is additive, thus from (28) we obtain

$$x[f_1(-x) + f_1(x)] = A_1(x)$$

and further

$$f_1(-x) + f_1(x) = \frac{A_1(x)}{x}, \quad x \neq 0.$$

Now we put $H(x) := \frac{f_1(x) - f_1(-x)}{2}$, which gives us

$$f_1(x) = \frac{f_1(x) - f_1(-x)}{2} + \frac{f_1(x) + f_1(-x)}{2} = H(x) + \frac{A_1(x)}{2x}$$

for all $x \in \mathbb{R} \setminus \{0\}$. And from the definition of f we get

$$f(x) = f_1\left(\frac{x}{\alpha}\right) = H\left(\frac{x}{\alpha}\right) + \frac{\alpha A_1\left(\frac{x}{\alpha}\right)}{2x}.$$

To finish this part of the proof it suffices to take $h(x) := H\left(\frac{x}{\alpha}\right)$ and $A(x) := \frac{\alpha}{2} A_1\left(\frac{x}{\alpha}\right)$.

Finally we consider the case $\alpha \neq -\beta$. Note that both numbers α, β are different from zero. Now we put in (2) $\frac{1}{\alpha}x$ and $\frac{1}{\alpha}y$ instead of x and y , respectively. Thus

$$F\left(\frac{1}{\alpha}y\right) - F\left(\frac{1}{\alpha}x\right) = \frac{1}{\alpha}(y - x) \left[f\left(x + \frac{\beta}{\alpha}y\right) + f\left(\frac{\beta}{\alpha}x + y\right) \right].$$

Multiplying this equation by α and taking $G(x) := \alpha F\left(\frac{1}{\alpha}x\right)$, $\gamma := \frac{\beta}{\alpha}$ we obtain that f satisfies

$$G(y) - G(x) = (y - x)[f(x + \gamma y) + f(\gamma x + y)].$$

Thus we have got the equation (12) which in view of Theorem 1 means that

$$f(x) = cx^3 + bx^2 + ax + d, \quad x \in \mathbb{R}.$$

To finish this part of the proof we assume that $c \neq 0$ and we are going to show that

$$\alpha^2 + \beta^2 = 4\alpha\beta.$$

Indeed, if function $f(x) = cx^3 + bx^2 + ax + d$ satisfies (2) then from Lemma 2 we know that also $f_3(x) := cx^3$ satisfies (2). Thus we have

$$G_3(y) - G_3(x) = (y - x)[f_3(\alpha x + \beta y) + f_3(\beta x + \alpha y)], \quad x, y \in \mathbb{R}$$

for some function G_3 . Using the form of f and equality

$$G_3(y) = y[f_3(\beta y) + f_3(\alpha y)], \quad y \in \mathbb{R},$$

we get after some simple calculations

$$(\beta^3 + \alpha^3)(y^4 - x^4) = (y - x)[(\alpha^3 + \beta^3)(x^3 + y^3) + 3\alpha\beta(\alpha + \beta)(xy^2 + x^2y)],$$

i.e.

$$(\beta^3 + \alpha^3)(y^3 + x^3 + xy^2 + x^2y) = (\alpha^3 + \beta^3)(x^3 + y^3) + 3\alpha\beta(\alpha + \beta)(xy^2 + x^2y),$$

and further

$$(\beta^3 + \alpha^3) = 3\alpha\beta(\alpha + \beta),$$

thus

$$4\alpha\beta = \alpha^2 + \beta^2.$$

Similarly, if we assume that $b \neq 0$ we obtain the same equality for α and β .

On the other hand, it is easy to show that in every of the above cases obtained functions satisfy equation (2). \square

REMARK 2. Let us consider the following quadrature rule with two nodes

$$\int_x^y f(t)dt \approx \frac{1}{2}(y - x)[f(\alpha x + (1 - \alpha)y) + f((1 - \alpha)x + \alpha y)],$$

which leads to a functional equation

$$F(y) - F(x) = (y - x)[f(\alpha x + (1 - \alpha)y) + f((1 - \alpha)x + \alpha y)].$$

If we additionally assume that this equation is satisfied by $f(x) = x^3$ then

$$\alpha = \frac{3 - \sqrt{3}}{6}.$$

This means that from our result we obtain that the only two-nodes quadrature rule satisfied exactly by polynomials of degree 3 (or 2) is the well known Gauss quadrature rule. One should emphasize that we assume here no regularity of functions f, F .

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